Supplementary Note 1: Finite Temperature Hartree-Fock Calculation of the Quantum Capacitance

The dashed green curve in figure 5 is obtained from an approximate finite temperature Hartree-Fock calculation of the quantum capacitance of a nanotube that takes into account the first two subbands. Following ref. 3 we assume a Coulomb potential $V(x) = \frac{e^2}{4\pi\varepsilon}\sqrt{x^2 + R^2}$ that is constant along the radial direction of the nanotube (*R* is the radius of the tube, *x* the coordinate along its axis, and ε is the dielectric constant of SiO_x). The exchange potential for electrons in a given subband is then independent of the electron density in the other subband. In addition, we neglect the momentum-dependence of the exchange potential and approximate it by a suitable average over momenta. In the low-density limit, when the chemical potential lies well below the bottom E_j of the j^{th} subband, $E_j - \mu >> kT$, we weigh the exchange potentials in this average with the occupation numbers of the corresponding states. In a quadratic approximation of the electronic dispersion relation appropriate at these low chemical potentials we then find that the exchange processes effectively modify the chemical potential by the amount:

$$\mu_{x,j}^{low-n} = -\frac{e^2}{4\pi\varepsilon}\frac{n_j}{N}\left[\ln\left(\frac{L_j}{R}\right)^2 + c\right] \qquad \text{(S1)}$$

Here $c = 2 - 2\gamma$, γ is the Euler constant, n_j is the density of the j^{th} subband, $N = 4$ is its degeneracy, m^*_j is its effective mass, and $L_j = 2\eta e^{\gamma/2-1}/\sqrt{m^*_j kT}$ is a thermal length.

At intermediate densities we interpolate between this expression and the corresponding exchange potential at zero temperature obtained in ref. 3 that describes the high density limit when $\mu - E_j >> kT$ and use the approximate exchange potential

$$\mu_{x,j}^{\text{int}} = -\frac{e^2}{4\pi\varepsilon}\frac{n_j}{N}\left[\ln\frac{\left(N/\pi n_j\right)^2 + L_j^{\,2}}{R^2} + c\right]. \qquad \text{(S2)}$$

Screening by a nearby gate has a similar effect on the exchange at low densities. The obtained cutoff length in our device is larger than the thermal length and has therefore a negligible effect.

At a given μ the carrier densities in the two relevant subbands of the nanotube are obtained from the Fermi-Dirac distribution as:

$$n_j(\mu) = N\int dE\,\nu_j(E)\frac{1}{1+\exp\left[\left(E+\mu_{x,j}^{\text{int}}-\mu\right)/kT\right]}, \qquad \text{(S3)}$$

where $\nu_j(E)$ is the single-particle density of states of the j^{th} subband.

Solving self-consistently equations S2 and S3 at a given total density, $n = n_1 + n_2$, under the assumption that the Coulomb potential prohibits spatial density fluctuations we obtain the dependence of the chemical potential on the total density, $\mu(n)$. From its derivative we finally find the quantum contributions to the capacitance in the Hartree-Fock approximation shown in figure 5:

$$C_{dos}^{-1} + C_x^{-1} = e^{-2}d\mu/dn \qquad \text{(S5)}$$